# BOUNDS FOR EFFECTIVE ELASTIC MODULI OF INHOMOGENEOUS SOLID BODIES

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In connection with the extensive use of various kinds of inhomogeneous materials (glass, carbon and boron reinforced plastics, cermets, concrete, reinforced materials, etc.) in technology, there arises a need to calculate the elastic properties of such systems. Here in each case it is necessary to work out specific methods for finding both elastic fields and effective moduli. Since, as a rule, such methods do not take into account the character of distribution of inhomogeneities in space, which is reflected on the form of the central moment functions [1], they can be referred to a single class and, consequently, can be obtained by a common method [2]. In the given paper, by means of the method of solution of stochastic problems for microinhomogeneous solid bodies proposed in the work of the author [2], we find elastic fields and effective moduli in an arbitrary approximation. Depending on the choice of parameters, the latter form bounds within which there lie the exact values of the effective moduli. It is shown that the conditions used earlier for finding these parameters [3] are not the best ones. The effective elastic moduli of an inhomogeneous medium are calculated, and bounds, narrower than the bounds formed in [3], are found for them.

1. Let the elastic properties of the statistically homogeneous infinite medium under consideration be described by a random tensor field $\lambda_{ijkl}(\mathbf{r})$. Side by side with this we introduce for comparison a homogeneous tensor field $\lambda^{c}_{ijkl}$ which characterizes the elastic properties of a certain homogeneous body.

The fields of displacements $u_i$ and $u_i^c$, corresponding to the two tensors of the elastic moduli, satisfy the equations

$$L_{ik}u_k = -f_i, \quad L_{ik} = \nabla_j \lambda_{ijkl} \nabla_l$$
$$L^c_{ik}u^c_k = -f_i, \quad L^c_{ik} = \nabla_j \lambda^c_{ijkl} \nabla_l$$

where $f_i$ is the density vector of body forces.

The problem consists of finding the tensors of strains $\varepsilon_{ij} = 1/2\,(\nabla_i u_j + \nabla_j u_i) \equiv u_{(i,j)}$ and effective moduli $\lambda^{*}_{ijkl}$ which determine the mean strains $\langle \varepsilon_{ij} \rangle$ by means of the equation

$$L^{*}_{ik}\langle u_k \rangle = -f_i, \quad L^{*}_{ik} = \nabla_j \lambda^{*}_{ijkl} \nabla_l \tag{1.1}$$

Here the angle parentheses denote averaging over the region v whose dimensions are less than the scale of inhomogeneity of the regular component of the field $\langle \varepsilon_{ij} \rangle$, but which is much greater than the dimensions of a grain of inhomogeneity, under which we are to understand a region of constant value of $\lambda_{ijkl}$. For ergodic fields, averaging over the volume coincides with averaging over an ensemble of realizations.

In the general case the tensor $\lambda^{*}_{ijkl}$ possesses nonlocality; this leads to an integral connection between stresses and strains, or to necessity of taking into account the inhomogeneity of macroscopic fields of strain $\langle \varepsilon_{ij} \rangle$ [4]. However, when considering quasi-homogeneous fields $\langle \varepsilon_{ij} \rangle$, for which the dimensions of the region of inhomogeneity substantially exceed the scale of nonlocality $\lambda^{*}_{ijkl}$, this nonlocality does not manifest itself [5-7] and the quantity $\lambda^{*}_{ijkl}$ in Eq. (1.1) can be considered as a usual tensor.

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It can be shown [3] that for these fields $\varepsilon_{ij}^{c} = \langle \varepsilon_{ij} \rangle$. Here the function $f_1$ obviously has a region of inhomogeneity of the same order as the field $\langle \varepsilon_{ij} \rangle$.

Omitting in the following the tensor indices and using the results of Sections 1 and 2 of [2], we write the expression for the strain field,

$$\varepsilon = \lim_{n\to\infty} \varepsilon_n, \quad \varepsilon_n = A_n \langle \varepsilon \rangle, \quad \lambda' = \lambda - \lambda_c$$
$$A_n = (1 - g\lambda')^{-1} R_n \langle (1 - g\lambda')^{-1} R_n \rangle^{-1}, \quad R_n = \sum_{k=0}^{n} (Hl)^k \tag{1.2}$$

where the operator H operates according to the rule

$$(Hl)^k = Hl\,(Hl)^{k-1} = hl\,(Hl)^{k-1} - \langle hl\,(Hl)^{k-1} \rangle \equiv \delta\,[hl\,(Hl)^{k-1}]$$

while $\delta$ is the operator of taking the random component. Here g and h are the operators determined, respectively, by the singular and formal components of the second derivative of the Green tensor of the operator $L_c$.

With (1.2) taken into account, the corresponding approximation for the stress field and the tensor of effective moduli has the form

$$\sigma = \lim_{n\to\infty} \sigma_n, \quad \sigma_n = \lambda A_n \langle \varepsilon \rangle, \quad \langle \sigma_n \rangle = \lambda_n \langle \varepsilon \rangle$$
$$\lambda_* = \lim_{n\to\infty} \lambda_n, \quad \lambda_n = \langle \lambda A_n \rangle \tag{1.3}$$

The expressions (1.2) and (1.3) completely solve the problem of describing an inhomogeneous elastic medium in the n-th approximation. In the case $n \to \infty$ we obtain the exact solution. However, because of mathematical difficulties [2, 7, 10] in a majority of cases we have to confine ourselves to the zeroth (singular) approximation in the expression (1.2) and (1.3), which takes into account only the local interactions between grains of inhomogeneity

$$R_s \equiv R_0 = 1, \quad \lambda_s \equiv \lambda_0 = \langle \lambda A_0 \rangle \tag{1.4}$$
$$A_0 = X^{-1} X_c \langle X^{-1} X_c \rangle^{-1}, \quad X = \lambda + b_c, \quad X_c = \lambda_c + b_c \tag{1.5}$$

while the tensor $b_c$ is defined by the equation $gX_c = -1$.

It should be noted that the statistical averaging used here presupposes averaging both with respect to the realizations of the elastic moduli and with respect to the realizations of the form of grains of inhomogeneity [11]. The latter is conveniently described by a vector **a** drawn from the origin of the coordinates, located at the center of mass of a homogeneous grain, to a point lying on the surface bounding it. The tensor g is a function of the shape of the grain surface and, consequently, of the vector **a**.

The expressions (1.4)and (1.5) thus allow us to calculate the effective moduli in the case of an arbitrary microinhomogeneous medium in a singular approximation. If the grains have a spherical shape or their orientations are strictly ordered (a full mechanical texture), then averaging with respect to the form of grains (or with respect to the realizations g) is dropped. In this case from (1.5) and (1.4) we have

$$X_s^{-1} = \langle X^{-1} \rangle, \quad X_s = \lambda_s + b_c \tag{1.6}$$

From (1.6), having carried out averaging, we can obtain an explicit form of effective elastic moduli $\lambda_s$ which will depend on the parameter $\lambda_c$ and the shape of the grain. A solution coinciding with (1.6) was obtained in [2, 3, 7-14], but the form of the grain (polycrystalline, mechanical mixture), the shape of the grain, as well as the value of the parameter $\lambda_c$ in these investigations were different.

2. We shall now establish the connection between $\lambda_*$ and its approximate value $\lambda_n$ in the form of an inequality whose sign is determined by the value of $\lambda_c$. With this aim we consider the doubled density of potential energy of elastic strains $\varepsilon\lambda\varepsilon$. Its average value over a characteristic volume v, if we recall Section 1, in view of the quasi-homogeneity of the field $\langle \varepsilon \rangle$ satisfies the following equations:

$$\langle \varepsilon\lambda\varepsilon \rangle = \langle \varepsilon \rangle \lambda_* \langle \varepsilon \rangle = \frac{1}{v} \int_v \varepsilon\lambda\varepsilon\, dV = \frac{1}{v} \int_v \langle \varepsilon\lambda\varepsilon \rangle\, dV \tag{2.1}$$

Taking into account (2.1), we write the total potential energy of strain U in the form

$$U = \frac{1}{2}\int \varepsilon\lambda\varepsilon \, dV = \frac{1}{2}\int \langle\varepsilon\lambda\varepsilon\rangle \, dV$$

where integration is carried out over the entire space.

In the first boundary value problem a displacement is specified on the surface of the region. According to the minimum principle of potential energy [14] the total strain energy Y in the unique solution ($\sigma$, $\varepsilon$) will be less than in any other virtual strains $\tilde{\varepsilon}$ which correspond to the field of displacements, continuously and piecewise continuously differentiable, assuming given values on the surface.

Thus, the inequality [14]

$$U \leqslant \frac{1}{2}\int \langle\tilde{\varepsilon}\lambda\tilde{\varepsilon}\rangle \, dV$$

or the inequality

$$\langle\varepsilon\lambda\varepsilon\rangle \leqslant \langle\tilde{\varepsilon}\lambda\tilde{\varepsilon}\rangle \tag{2.2}$$

reducible to it is valid.

Adding to (2.2) the equation

$$0 = \langle(\langle\varepsilon\rangle - \tilde{\varepsilon})\,\tilde{\sigma}\rangle$$

which is valid for quasi-homogeneous fields, we obtain

$$\langle\varepsilon\rangle\lambda_*\langle\varepsilon\rangle \leqslant \langle\varepsilon\rangle\langle\tilde{\sigma}\rangle + \langle\tilde{\varepsilon}\ (\lambda\tilde{\varepsilon}\ \tilde{\sigma}) - \rangle \tag{2.3}$$

Here $\sigma$ is an arbitrary piecewise continuous differentiable field of stresses which satisfies the equation

$$\tilde{\sigma}_{ij,j} = -f_i \tag{2.4}$$

The representation of $\tilde{\sigma}$ which is usual for problems of this kind has the form [3, 12-14]

$$\tilde{\sigma} = \lambda_c\tilde{\varepsilon} + \tilde{\tau} \tag{2.5}$$

where the polarized stress $\tilde{\tau}$ in accordance with (2.4) satisfies the equation

$$L_{ik}{}^c\tilde{u}_k + \tilde{\tau}_{ik,k} = -f_i \tag{2.6}$$

Substracting from (2.6) the equation obtained by its averaging, we find the connection between $\tilde{\varepsilon}$ and $\tilde{\tau}$

$$\tilde{\varepsilon} = \langle\varepsilon\rangle + G * \delta\tilde{\tau} \tag{2.7}$$

Here G is the second derivative of the Green tensor of the operator $L_C$. The form of the tensor $\tilde{\tau}$ whose properties are described in detail in [14] influences $\tilde{\sigma}$, where in the case $\tilde{\tau} = \lambda'\tilde{\varepsilon}$ the stress $\tilde{\sigma}$ and the strain $\tilde{\varepsilon}$ coincide with the true values $\sigma$ and $\varepsilon$.

We choose the approximating value $\tilde{\tau}$ in the form

$$\tilde{\tau} = \lambda'\varepsilon_n \tag{2.8}$$

where $\varepsilon_n$ is determined according to (1.2). Then from (2.5) we have

$$\tilde{\sigma} = \lambda\varepsilon_n + \lambda_c(\tilde{\varepsilon} - \varepsilon_n) \tag{2.9}$$

Substituting (2.9) in (2.3) and taking into account (2.5) and (2.8), we obtain

$$\begin{gathered}\langle\varepsilon\rangle\lambda_*\langle\varepsilon\rangle \leqslant \langle\varepsilon\rangle\lambda_c\langle\varepsilon\rangle + \langle\varepsilon\rangle\langle\tilde{\tau}\rangle + \langle\tilde{\tau}(\tilde{\varepsilon} - \varepsilon_n)\rangle + M_1 \\ M_1 = \langle(\tilde{\varepsilon} - \varepsilon_n)\lambda'(\tilde{\varepsilon} - \varepsilon_n)\rangle\end{gathered} \tag{2.10}$$

Let the inequality

$$M_1 \leqslant 0 \tag{2.11}$$

hold; the conditions of fulfilment of this inequality are discussed below. Then from (2.10) we find

$$\langle\varepsilon\rangle\lambda_*\langle\varepsilon\rangle \leqslant \langle\varepsilon\rangle\lambda_c\langle\varepsilon\rangle + \langle\varepsilon\rangle\langle\tilde{\tau}\rangle + \langle\tilde{\tau}(\tilde{\varepsilon} - \varepsilon_n)\rangle \tag{2.12}$$

The random field $\tilde{\varepsilon}-\varepsilon_n$ by means of (2.7) and (2.8) can be brought into the form

$$\tilde{\varepsilon}-\varepsilon_n=\langle\varepsilon\rangle+G_* \delta\tilde{\tau}-p\tilde{\tau}, \quad p\lambda'=1 \tag{2.13}$$

Substitution of (2.13) into (2.12) gives

$$\langle\varepsilon\rangle\lambda_*\langle\varepsilon\rangle\leqslant\langle\varepsilon\rangle\lambda_c\langle\varepsilon\rangle+2\langle\varepsilon\rangle\langle\tilde{\tau}\rangle-\langle\tilde{\tau}p\tilde{\tau}\rangle+\langle\delta\tilde{\tau}G_*\delta\tilde{\tau}\rangle \tag{2.14}$$

It can be shown [3] that the right side (2.14) has an extremum under the condition

$$\tilde{\varepsilon}=p\tilde{\tau}$$

which in accordance with (2.8) gives

$$\tilde{\varepsilon}=\varepsilon_n \tag{2.15}$$

When the inequality (2.11) is fulfilled, this extremum will be a minimum [3].

By means of (1.2), (1.3), (2.8), and (2.15) from (2.12) we obtain the inequality

$$\langle\varepsilon\rangle\lambda_*\langle\varepsilon\rangle\leqslant\langle\varepsilon\rangle\langle\lambda\varepsilon_n\rangle=\langle\varepsilon\rangle\lambda_n\langle\varepsilon\rangle$$

which establishes the upper bound for $\lambda_*$

$$\lambda_*\leqslant\lambda_n \tag{2.16}$$

If instead of (2.11) the inequality

$$N_1=\langle\lambda_c(\tilde{\varepsilon}-\varepsilon_n)s'\lambda_c(\tilde{\varepsilon}-\varepsilon_n)\rangle\leqslant 0 \quad (s'=s-s_c) \tag{2.17}$$

is fulfilled, where s is the compliance tensor, inverse to the tensor $\lambda$, then, using the theorem of minimum complementary energy [14], we obtain

$$\lambda_*\geqslant\lambda_n \tag{2.18}$$

which gives the lower bound.

We must bear in mind that in view of the conditions (2.11) and (2.17) the tensor $\lambda_c$, determining $\lambda_n$, turns out to be different for (2.16) and (2.18). Denoting by $\lambda_c\pm$ the tensor $\lambda_c$ satisfying respectively the inequalities (2.11) and (2.17), from (2.16) and (2.18) we find

$$\lambda_n^-\leqslant\lambda_*\leqslant\lambda_n^+ \tag{2.19}$$

where $\lambda_n\pm$ are the values of the tensor $\lambda_n$ obtained by means of the tensors $\lambda_c\pm$ respectively.

Carrying out an analogous analysis for the second boundary value problem, when loads are specified on the surface [14], we find bounds for $\lambda_*$ in the form (2.19). Here $\lambda_c\pm$ satisfy the inequalities

$$\begin{aligned}M_2&=\langle s_c(\tilde{\sigma}-\sigma_n)\lambda' s_c(\tilde{\sigma}-\sigma_n)\rangle\leqslant 0\\N_2&=\langle(\tilde{\sigma}-\sigma_n)s'(\tilde{\sigma}-\sigma_n)\rangle\leqslant 0\end{aligned} \tag{2.20}$$

which respectively replace the inequalities (2.11) and (2.17).

It should be noted that the bounds (2.19), in contrast to the bounds obtained in [3], to which the zeroth approximation corresponds, can be made arbitrarily narrow, and for $n\to\infty$ they coincide with the exact value of the effective moduli $\lambda_*$. To calculate $\lambda_n$, however, we need information about the central moment functions of higher orders [1].

3. Concluding, we show that the bounds (2.19) can be improved in comparison with the bounds of Hashin and Shtrikman [3] also as a result of a better choice of the parameter $\lambda_c$. In [3, 12-14] it is assumed that the inequalities (2.11), (2.17), and (2.20) take place under the condition

$$\lambda'\leqslant 0 \tag{3.1}$$

for the forms $M_1$ and $M_2$ and

$$s'\leqslant 0 \tag{3.2}$$

for the forms $N_1$ and $N_2$. It is obvious that the latter is equivalent to the inequality

$$\lambda'\geqslant 0 \tag{3.3}$$

The inequalities (3.1)-(3.3) should be understood, as usual, in the sense of negative (positive) semi-definiteness of quadratic forms set up by means of the tensors $\lambda'$ and $s'$.

However, (3.1) and (3.2) are not the only possible solutions of the inequalities (2.11), (2.17), and (2.20). We shall consider, in particular, the singular approximation for a case where body forces are absent, while the macroscopic fields are homogeneous.

Then the quadratic forms M and N reduce to moment functions of the third order, we can let the volume v tend to infinity, and the elastic fields of stresses and strains are homogeneous within the grain [12, 13].

It is easy to see that in the case of a mechanical mixture of two isotropic components the inequalities $M_1 \leq 0$ and $M_2 \leq 0$ are satisfied under the condition

$$c_1\lambda_2 + c_2\lambda_1 - \lambda_c \leqslant 0 \tag{3.4}$$

while the inequalities $N_1 \leq 0$ and $N_2 \leq 0$ are satisfied under the condition

$$c_1 s_2 + c_2 s_1 - s_c \leqslant 0 \tag{3.5}$$

where $c_\alpha$, $\lambda_\alpha$, and $s_\alpha$ are the volume concentration and the tensors of the elastic moduli and elastic compliances of the $\alpha$-th component. As both components are isotropic, for a macroscopically isotropic medium the comparison field $\lambda_c$ must also be chosen isotropic. But then inequalities (3.4) and (3.5) determine the tensors

$$\lambda_c^+ = \lambda_1 \langle \lambda^{-1} \rangle \lambda_2, \quad \lambda_c^- = \lambda_1 \langle \lambda \rangle^{-1} \lambda_2 \tag{3.6}$$

which for $\lambda_1 \leq \lambda_2$ satisfy the inequalities

$$\lambda_1 \leqslant \lambda_c^- \leqslant \lambda_c^+ \leqslant \lambda_2 \tag{3.7}$$

Since according to (3.1)-(3.3) in the role of $\lambda_c^+$ and $\lambda_c^-$ we must choose $\lambda_2$ and $\lambda_1$ [1], while $\lambda_s$ increases with $\lambda_c$ [12], the chosen $\lambda_c^\pm$ leads to narrower bounds. Denoting by $\lambda_H^\pm$ the bounds found by means of $\lambda_c^\pm$ which satisfy (3.1)-(3.3) [3, 12-14], while by $\lambda_s^\pm$ denoting the analogous values obtained by means of (3.6), and taking into account (3.7), we write

$$\lambda_H^- \leqslant \lambda_s^- \leqslant \lambda_* \leqslant \lambda_s^+ \leqslant \lambda_H^-$$

Thus, use of all the information contained by the quadratic forms M and N enables us, even within the framework of the singular approximation, to narrow the bounds of Hashin and Shtrikman as a result of a better choice of the parameter $\lambda_c$. A further narrowing of the bounds is possible only if we take into account inhomogeneities of the field in the grain. This field for $n \neq 0$ in (1.2) and (1.3) is described by terms obtained by means of the nonlocal operator h.

Concluding, we note that the average values of effective moduli obtained in [7-10] are equivalent to the values of $\lambda_s$ calculated by means of (1.6), if in the role of $\lambda_c$ we choose the value $\lambda_V \equiv \langle \lambda \rangle$ for the so-called Voigt model [7-10] and $\lambda_R \equiv \langle \lambda^{-1} \rangle^{-1}$ for the Reuss model [7, 10]. Although in both cases the values of $\lambda_s$ thus found lie within the bounds of Hashin and Shtrikman [7], they themselves do not form bounds. Indeed, the values of $\lambda_V$ and $\lambda_R$ satisfy the inequality (3.4) under the condition

$$(c_1 - c_2)\ (\lambda_1 - \lambda_2) \geqslant 0 \tag{3.8}$$

$$(c_1\lambda_2 + c_2\lambda_1)^2 \leqslant \lambda_1\lambda_2 \tag{3.9}$$

which for certain concentrations are simultaneously fulfilled. In this case both the Voigt and Reuss models give the upper bounds. On the other hand, $\lambda_V^{-1}$ and $\lambda_R^{-1}$ satisfy the inequality (3.5) under the conditions

$$(c_1\lambda_1 + c_2\lambda_2)^2 \leqslant \lambda_1\lambda_2 \tag{3.10}$$

$$(c_1 - c_2)\ (\lambda_1 - \lambda_2) \leqslant 0 \tag{3.11}$$

When (3.10) and (3.11) are simultaneously fulfilled, both models give the lower bounds. However, for the values of effective moduli obtained in the Voigt model ($\lambda_c = \lambda_V$) to form bounds for $\lambda_*$, simultaneous fulfilment of the inequalities (3.8) and (3.11) is necessary.

Since this is impossible, solutions in the Voigt and Reuss models do not lead to setting up of the bounds, but can be used to improve one of them.

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